

Multiplicity of positive solutions for a critical quasilinear Neumann problem

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Abstract

We establish the multiplicity of positive solutions to a quasilinear Neumann problem in expanding balls and hemispheres with critical exponent in the boundary condition.

1 Introduction

We consider the following problem

$$\begin{cases} \Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = |u|^{p-2} u & \text{in } B_R, \\ |\nabla u|^{p-2} \langle \nabla u; \mathbf{n} \rangle = |u|^{q-2} u & \text{on } S_R, \\ u > 0 & \text{in } B_R, \end{cases} \quad (1)$$

where B_R and S_R are the ball and the sphere with radius R respectively in \mathbb{R}^n . Here $1 < p < n$ and $q = p^{**} = \frac{(n-1)p}{(n-p)}$ is the critical exponent for the trace embedding.

We establish the multiplicity effect for weak solutions to (1). Namely we prove that the number of positive rotationally non-equivalent solutions is unbounded as $R \rightarrow \infty$.

The effect of multiplicity was discovered by Coffman [5] who considered the Dirichlet problem

$$\begin{cases} -\Delta_p u = |u|^{q-2} u & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega_R, \\ u > 0 & \text{in } \Omega_R, \end{cases} \quad (2)$$

where Ω_R is the annulus $B_R \setminus B_{R-1} \subset \mathbb{R}^n$ for $n = 2$ and $p = 2$. The problems (1) and (2) were studied later by many authors for subcritical q (see e.g. [17, 8, 10, 11, 4]). In [20] the multiplicity result was obtained for the Neumann problem

$$\begin{cases} -\Delta u + \lambda u = |u|^{p^*-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω satisfies some symmetry conditions and p^* is the critical exponent for the Sobolev trace embedding.

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One can easily show that after suitable rescaling solutions of (1) are solutions to the following problem:

$$\begin{cases} \Delta_p u = \lambda |u|^{p-2} u & \text{in } B, \\ |\nabla u|^{p-2} \langle \nabla u; \mathbf{n} \rangle = |u|^{q-2} u & \text{on } S, \\ u > 0 & \text{in } B, \end{cases} \quad (3)$$

where $B = B_1$, $S = S_1$ and $\lambda(R) = R^p$ as $R \rightarrow \infty$.

We look for distinct solutions of the problem (3) by minimizing the functional

$$I^\lambda[u] := \frac{\|\nabla u\|_{L_p(B)}^p + \lambda \|u\|_{L_p(B)}^p}{\|u\|_{L_q(S)}^p} \quad (4)$$

on different subsets of $W_p^1(B)$.

In order to construct solutions to problem (3) let us introduce the following notation:

Definition 1. Let $A \subset S$ and $\varkappa > 0$. We denote by A^δ \varkappa -neighborhood of a set A , i.e.

$$A^\delta = \{z \in S \mid \text{dist}(z, A) \leq \delta\}.$$

The following definition was introduced in [4]:

Definition 2. Let G be a closed subgroup of $O(n)$. We call set $A \subset S$ a locally minimal orbital set under the action of G if A is invariant under the action of G and satisfies the following conditions:

- for any $x \in A$ the orbit Gx is a discrete set and $m(A) := |Gx|$ is independent of x .
- there exists $\varkappa > 0$ such that for any $y \in A^\varkappa \setminus A$ and $x \in A$, we have $|Gx| < |Gy|$.

We denote as $m(G)$ the number of elements in the minimal orbit of G and $K(n, p)$ stands for the best Sobolev trace constant in half-space defined as

$$K(n, p) = \inf_{v \in C_c^\infty(\mathbb{R}_+^n) \setminus \{0\}} \frac{\|\nabla v\|_{L_p(\mathbb{R}_+^n)}^p}{\|v(\cdot, 0)\|_{L_q(\mathbb{R}^{n-1})}^p}.$$

The value of $K(n, p)$ is calculated explicitly in [9] for $p = 2$ and [13] for arbitrary p .

We consider local minimizers of functional (4) on sets

$$X_G(A, \beta) = \{u \in W_p^1(B) \mid u(gx) \equiv u(x) \ \forall g \in G, \|u\|_{L_q(S)} = 1, \|u\|_{L_q(A^\varkappa)}^q \geq 1 - \beta\},$$

where G is some closed subgroup of $O(n)$, A is a locally minimal orbital set and β is some small parameter that we will choose later. We denote $X_G(A, \beta)$ by X if it does not lead to confusion.

The structure of the paper is as follows. In Section 2 we prove some auxiliary lemmas and in Section 3 we establish main multiplicity results.

2 Auxiliary lemmas

The following fact is well known and will be given here without a proof.

Proposition 1. *The functional $I^\lambda[u]$ is Gateaux differentiable and for any $h \in W_p^1(B)$*

$$\begin{aligned} DI^\lambda[u](h) &= p \int_B |\nabla u|^{p-2} \nabla u \cdot \nabla h \, dx \frac{1}{\|u\|_{L_q(S)}^p} - p \int_B |\nabla u|^p \, dx \int_S |u|^{q-2} u h \, dS \frac{1}{\|u\|_{L_q(S)}^{p+q}} \\ &\quad - p\lambda \int_B |u|^p \, dx \int_S |u|^{q-2} u h \, dS \frac{1}{\|u\|_{L_q(S)}^{p+q}} + p\lambda \int_B |u|^{p-2} u h \, dx \frac{1}{\|u\|_{L_q(S)}^p}. \end{aligned}$$

Lemma 1. *Let $u_j^\lambda \in W_p^1(B)$ be a bounded Palais-Smale sequence for I^λ at the level $c > 0$. Then there is $u_0^\lambda \in W_p^1(B)$ such that up to subsequence $u_j^\lambda \rightharpoonup u_0^\lambda$ and*

$$|\nabla u_j^\lambda|^p \, dx \rightharpoonup \mu \geq |\nabla u_0^\lambda|^p \, dx + \sum_k \mu_k \delta(x - x_k), \quad (5)$$

$$|u_j^\lambda|^q \, dS \rightharpoonup \nu = |u_0^\lambda|^q \, dS + \sum_k \nu_k \delta(x - x_k), \quad (6)$$

where $\delta(x - x_k)$ are delta measures at some points x_k in S and $\mu_k \geq K(n, p) \nu_k^{\frac{p}{q}}$. Furthermore, either $\nu_k = 0$ or $\nu_k \geq (c^{-1} \cdot K(n, p))^{\frac{q}{q-p}} \nu(S)$.

Proof. Since $\{u_j^\lambda\}$ is bounded in $W_p^1(B)$, the relations (5) and (6) follow by the Lions concentration-compactness principle [12]. Since I^λ is homogeneous we can assume without loss of generality that $\|u_j^\lambda\|_{L_q(S)} = 1$ and $\nu(S) = 1$. Next we use the argument from [6, 1]: Let us fix x_k from (5) and (6). We choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\varphi = 1 \text{ in } B(x_k, \varepsilon), \quad \varphi = 0 \text{ in } \mathbb{R}^n \setminus B(x_k, 2\varepsilon), \quad |\nabla \varphi| \leq \frac{C}{\varepsilon}.$$

Since $DI^\lambda[u_j^\lambda] \rightarrow 0$ we obtain

$$\lim_{j \rightarrow \infty} \langle DI^\lambda[u_j^\lambda], \varphi u_j^\lambda \rangle = 0$$

Then

$$\lim_{j \rightarrow \infty} \int_B |\nabla u_j^\lambda|^{p-2} \nabla u_j^\lambda \cdot \nabla \varphi u_j^\lambda \, dx = c \int_S \varphi \, d\nu - \int_B \varphi \, d\mu - \lambda \int_B |u_0^\lambda|^p \varphi \, dx. \quad (7)$$

One can estimate the left hand side as follows:

$$\begin{aligned} 0 &\leq \left| \lim_{j \rightarrow \infty} \int_B |\nabla u_j^\lambda|^{p-2} \nabla u_j^\lambda \cdot \nabla \varphi u_j^\lambda \, dx \right| \\ &\leq \lim_{j \rightarrow \infty} \left(\int_B |\nabla u_j^\lambda|^p \, dx \right)^{\frac{p-1}{p}} \left(\int_B |\nabla \varphi|^p |u_j^\lambda|^p \, dx \right)^{\frac{1}{p}} \\ &\leq C \left(\int_{B(x_k, 2\varepsilon)} |\nabla \varphi|^p |u_0^\lambda|^p \, dx \right)^{\frac{1}{p}} \\ &\leq C \left(\int_{B(x_k, 2\varepsilon)} |\nabla \varphi|^n \, dx \right)^{\frac{1}{n}} \left(\int_{B(x_k, 2\varepsilon)} |u_0^\lambda|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-p}{pn}} \\ &\leq C \left(\int_{B(x_k, 2\varepsilon)} |u_0^\lambda|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-p}{pn}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Taking the limit in (7) we get $\nu_k = c^{-1}\mu_k \geq c^{-1}K(n,p)\nu_k^{\frac{p}{q}}$. This means either $\nu_k \geq (c^{-1}K(n,p))^{\frac{q}{q-p}}$ or $\nu_k = 0$. \square

Lemma 2. *Let $u^\lambda \in X$ be a sequence such that $I^\lambda[u^\lambda] \leq K(n,p)m(A)^{1-\frac{p}{q}}$. Then there is a $\beta_0 > 0$ such that for any $\beta \leq \beta_0$ there is $x_0 \in S$ such that we have up to subsequence the following weak convergence in the sense of measures as $\lambda \rightarrow \infty$:*

$$|u^\lambda|^q dS \rightharpoonup \sum_{x_k \in Gx_0} \frac{1}{m(A)} \delta(x - x_k). \quad (8)$$

Proof. Since $\|u^\lambda\|_{W_p^1}^p \leq I^\lambda[u^\lambda] \leq K(n,p)m(A)^{1-\frac{p}{q}}$ by the Lions concentration-compactness principle we get

$$\begin{aligned} |\nabla u^\lambda|^p dx &\rightharpoonup \mu \geq |\nabla u_0|^p dx + K(n,p) \sum_k \nu_k^{\frac{p}{q}} \delta(x - x_k), \\ |u^\lambda|^q dS &\rightharpoonup \nu = |u_0|^q dS + \sum_k \nu_k \delta(x - x_k), \end{aligned}$$

where $\delta(x - x_k)$ are delta measures at some points x_k in S .

Since $\lambda \|u^\lambda\|_{L_p(B)}^p$ is uniformly bounded, we have $u^\lambda \rightarrow 0$ in $L_p(B)$ so $u_0 = 0$. Combining the above with the fact that u^λ are invariant with respect to G we get:

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} I^\lambda[u^\lambda] &= \mu(B) \geq K(n,p) \sum_k \nu_k^{\frac{p}{q}} = K(n,p) \sum_j |Gx_j| \left(\frac{\tilde{\nu}_j}{|Gx_j|} \right)^{\frac{p}{q}} \\ &= K(n,p) \sum_j |Gx_j|^{1-\frac{p}{q}} \tilde{\nu}_j^{\frac{p}{q}}. \end{aligned} \quad (9)$$

Here j goes over different classes of equivalence of x_k , and $\tilde{\nu}_j = |Gx_j|\nu_j$ is a total contribution of that class to $\nu(\partial\Omega)$. The second equality is due to the fact that u^λ are G -invariant, so for every x_k there are $|Gx_k|$ δ -functions with the same coefficient.

Since $p < q$ we have $a^{\frac{p}{q}} + b^{\frac{p}{q}} > (a+b)^{\frac{p}{q}}$, for any $a > 0, b > 0$. Recalling that A is a locally minimal orbital set we can write

$$\begin{aligned} \mu(B) &\geq K(n,p)m(A)^{1-\frac{p}{q}} \sum_{j: x_j \in A} \tilde{\nu}_j^{\frac{p}{q}} + K(n,p) \sum_{i: x_i \notin A} |Gx_i|^{1-\frac{p}{q}} \tilde{\nu}_i^{\frac{p}{q}} \geq \\ &\geq K(n,p)(m(A)^{1-\frac{p}{q}} \alpha^{\frac{p}{q}} + m(G)^{1-\frac{p}{q}} (1-\alpha)^{\frac{p}{q}}), \end{aligned} \quad (10)$$

where $1 - \beta \leq \alpha \leq 1$ (we recall that $m(G)$ is the number of elements in the minimal orbit of G).

It's easy to see that the right hand side of (10) is a concave function of α . That means that if β is small enough, then the right hand side is a decreasing function, which achieves it's minimum of $K(n,p)m(A)^{1-\frac{p}{q}}$ at $\alpha = 1$.

Since by assumption $\mu(B) = \lim_{\lambda \rightarrow \infty} I^\lambda[u^\lambda] \leq K(n,p)m(A)^{1-\frac{p}{q}}$ we conclude that $\alpha = 1$. Recalling that for $u \in X$ $\|u\|_{L_q(S)} = 1$ we get (8). \square

From now on we always assume that λ is fixed and whenever there is a limit it is taken over $j \rightarrow \infty$ unless specified otherwise.

Lemma 3. *The minimum of I^λ on X is attained if λ is large enough and*

$$\inf_{u \in X} I^\lambda[u] < K(n, p)m(A)^{1-\frac{p}{q}}.$$

Proof. The Ekeland's variational principle [7] provides the existence of a minimizing sequence $u_j^\lambda \in X$ such that $I'[u_j^\lambda] \rightarrow 0$. Since u_j^λ is a Palais-Smale sequence at the level $\inf_{u \in X} I^\lambda[u] < K(n, p)m(A)^{1-\frac{p}{q}}$, Lemma 1 gives the estimate on any non-zero ν_k in (6):

$$\nu_k > m(A)^{-(1-\frac{p}{q})\frac{q}{q-p}} = \frac{1}{m(A)}. \quad (11)$$

Suppose that there is a δ -function outside of A . From (8) follows that for large λ almost all of $\nu(S)$ mass is concentrated in a \varkappa -neighbourhood of A , and according to (11) there are no δ -functions outside of that neighbourhood.

Let us suppose that there is a δ -function at $x_k \in A^\varkappa$ with weight ν_k . Since A is a locally minimal orbital set, we know that $|Gx_k| \geq m(A)$. Now from (9) and (11) we derive

$$\lim_{j \rightarrow \infty} I^\lambda[u_j^\lambda] \geq K(n, p)|Gx_k| \left(\frac{1}{m(A)} \right)^{\frac{p}{q}} = K(n, p)m(A)^{1-\frac{p}{q}}, \quad (12)$$

which is a contradiction.

From that follows that for u_0^λ in (8) $\|u_0^\lambda\|_{L_q(S)} = \|u_j^\lambda\|_{L_q(S)} = 1$. It is well known, that weak convergence and convergence of norms implies strong convergence in uniformly convex Banach space (e.g. [3, Proposition 3.32]), and that completes our proof. That way $u_0^\lambda \in X$ and $I^\lambda[u_0^\lambda]$ attains minimal value. \square

3 Main results

Lemma 4. *Let $G = H \times O(n - k)$ where H is a finite subgroup of $O(k)$ and $A \subset \mathbb{R}^k$ is a minimal orbital set under the action of H .*

Then for any fixed β , λ large enough and $p \leq \frac{n+1}{2}$ we have

$$\inf_{u \in X} I^\lambda[u] < K(n, p)m(A)^{1-\frac{p}{q}}. \quad (13)$$

Proof. Let $x_0 \in Gx_0$ be a point in $A \times \{0\}$. As was shown in [14] (see also [2]) there is a function u_R in $W_p^1(B_R)$ supported in a small ball around Rx_0 and axially symmetric with respect to the axis Ox_0 , such that $\|u_R\|_{W_p^1(B_R)}^p < K(n, p)\|u\|_{L_q(S_R)}^p$.

Now we construct the function

$$v_R(x) = \sum_{g \in H} u_R(gx).$$

It is easy to see that v_R is G -invariant and

$$\frac{\|v_R\|_{W_p^1(B_R)}^p}{\|v_R\|_{L_q(S_R)}^p} = m(A)^{1-\frac{p}{q}} \frac{\|u_R\|_{W_p^1(B_R)}^p}{\|u_R\|_{L_q(S_R)}^p} < K(n, p)m(A)^{1-\frac{p}{q}}.$$

By rescaling we obtain (13). \square

Theorem 1. *Let $p \leq \frac{n+1}{2}$ and let G be as in Lemma 4. Suppose that $A \subset \mathbb{R}^k$ is some locally minimal orbital set of H . Then there is $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ there is a G -invariant solution of problem (3) such that it concentrates at $|Gx_0|$ points in the Gx_0 for some $x_0 \in A \times \{0\}$, i.e.*

$$\frac{|u^\lambda|^q}{\|u^\lambda\|_{L_q(S)}^q} \rightharpoonup \sum_{k=1}^{|G(x_0)|} \frac{1}{|G(x_0)|} \delta(x - x_k) \quad \text{as } \lambda \rightarrow \infty.$$

Proof. According to Lemmas 4 and 3 there is a minimizer $u \in X$ such that it is concentrated around $m(A)$ points of $A \times \{0\}$. Lemma 2 implies that if λ is large enough the constraint $\|u\|_{L_q(A^\delta)}^q > 1 - \beta$ is non-active and does not produce a Lagrange multiplier. Since $I^\lambda[u] = I^\lambda[|u|]$ we can assume that u is non-negative. Since u is a local minimizer, we get for $\mu = I^\lambda[u]$ (see Proposition 1):

$$\int_B |\nabla u|^{p-2} \nabla u \cdot \nabla h \, dx + \lambda \int_B |u|^{p-2} u h \, dx - \mu \int_S |u|^{q-2} u h \, dS = 0 \quad \forall h \in L_G,$$

where

$$L_G = \{h \in W_p^1(B) \mid h(gx) = h(x) \, \forall g \in G\}.$$

Due to the principle of symmetric criticality [16] u is a solution to the problem

$$\begin{cases} \Delta_p u := \lambda |u|^{p-2} u & \text{in } B, \\ |\nabla u|^{p-2} \langle \nabla u; \mathbf{n} \rangle = \mu |u|^{q-2} u & \text{on } S, \end{cases}$$

Since $u \geq 0$ in B we can apply the Harnack inequality (see [19], [18]) and get the positivity of our solution. Since the boundary condition is not homogeneous, it's easy to show that $\mu^{\frac{1}{p-q}} u$ is a solution for problem (3). \square

Theorem 2. *For any $N > 0$ there is $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ problem (3) has at least N distinct solutions.*

Proof. Let us look at the following decomposition of \mathbb{R}^n :

$$\mathbb{R}^n = (\mathbb{R}^2)^l \times \mathbb{R}^m.$$

Here $l \geq 1$, $m \geq 0$. We denote variables in \mathbb{R}^n by x , in \mathbb{R}^2 by y and in \mathbb{R}^m by z . This way,

$$x = (y_1, y_2, \dots, y_l; z).$$

We introduce the group $G_{k,l} = H_{k,l} \times O(m)$ where $H_{k,l}$ is generated by rotations of every y_i by $\frac{2\pi}{k}$ and by transpositions of y_i and y_j for every i and j .

Let A be a globally minimal orbital set for the action of $H_{k,l}$. One can easily check that $A \times \{0\}$ is a locally minimal orbital set for $G_{k,l}$.

Now we show that for $l \geq 1$ and $k > 2$ the minimizers will be non-equivalent. In order to do that we analyse minimal orbits of $H_{k,l}$. The simple calculation yields that a minimal orbit would be of a point $(y, 0, \dots, 0) \in \mathbb{R}^{2l}$ where $y \in \mathbb{R}^2$ and it consists of $k \cdot l$ points. Knowing

the structure of the minimal orbits we can deduce that minimizers would be different for different pairs of (k, l) and (k', l') . \square

Now we consider an analogue of the problem (3) in an n -dimensional hemisphere.

To prove the multiplicity result we only need to modify lemma 4 by using the existence result from [15].

Lemma 5. *Let $n \geq 5$ and let B be an n -dimensional hemisphere. Let $G = H \times O(n - k)$ where H is a finite subgroup of $O(k)$ such that A is a minimal orbital set under the action of $H \times \{0\}$.*

Then for any fixed β , λ large enough and $2 < p \leq \frac{n+2}{3}$ we have

$$\inf_{u \in X} I^\lambda[u^\lambda] < K(n, p)m(A)^{1-\frac{p}{q}}.$$

Repeating the previous arguments we get the following theorem:

Theorem 3. *Let $n \geq 5$, and let B be an n -dimensional hemisphere, $2 < p \leq \frac{n+2}{3}$. Then for any $N > 0$ there is a $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ problem (3) has at least N rotationally non-equivalent solutions.*

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